Stationary soliton bound states existing in resonance with linear waves

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(Received 1 February 1995)

The phenomenon of stable propagation of spatially localized solitary waves, has been investigated for various dynamical systems. If a solitonlike pulse is in resonance with linear waves, then this pulse emits radiation and therefore it cannot exist as a stationary wave. Nevertheless, it is shown here that two (or more) radiating (and thus nonexisting as stationary waves) single solitonlike pulses can still form a *stationary* bound state due to mutual trapping of their own radiation. Trapped radiation forms a standing wave, which in turn produces local minima in an effective interaction potential of the neighboring solitons. However, in contrast to conventional solitons, soliton bound states that are formed due to trapped radiation exist only for discrete values of soliton parameters, i.e., such bound states do not form continuous families of localized solutions, and they are inherently unstable. Two physically important systems for which stationary bound states of radiating solitons can be found are considered.

PACS number(s): 03.40.Kf, 42.50.Rh, 02.30.Jr

I. INTRODUCTION

Various integrable dynamical equations, for example, the nonlinear Schrödinger (NLS) equation, the Korteweg–de Vries (KdV) equation, and the sine-Gordon (SG) equation, are known nowadays [1,2]. In physics these equations are in the center of interest because of their numerous applications [3]. One of the main features of these (and many other) integrable dynamical equations is the existence of *solitons*, stable spatially localized waves with the unique particlelike properties. In general, all solitons can be divided into two major classes: *single solitons* (which are stationary in a certain moving reference frame) and *breathers* (solitons localized in space and periodic in time which can be treated as *a nonlinear* superposition of single solitons).

In physical applications integrable dynamical equations usually correspond to certain approximations and therefore, in many cases, these equations should be generalized by some small additional terms to make the description of real physical systems more adequate. Thus the question of significant importance is what will happen with solitons of integrable equations if some perturbations are taken into account. (In this paper we are interested in *Hamiltonian* perturbations whereas the effects produced by non-Hamiltonian or *dissipative* perturbations seem to be well understood [4]. Also we will use the word "soliton" instead of "solitary wave" for localized solutions of nonintegrable equations as well.) Generally speaking, the answer to this question is known. Breather-type solitons are destroyed by almost any type of perturbations (e.g., see [5] for the SG breathers and [6,7] for the NLS breathers). However, single solitons have better chances to survive under action of Hamiltonian perturbations. In general, a single soliton is stable against perturbations that do not lead to resonant interactions of this soliton with small-amplitude linear waves [8]. In other words, the phase velocities of all spectral components of a soliton should not coincide with the phase velocity of any linear wave that can propagate in the system. The typical example illustrating this point is the generalized NLS equation with an additional third- [8–11,13] or fourth-order derivative term [12–16]:

$$i\frac{\partial U}{\partial \tau}+\frac{\partial^2 U}{\partial \zeta^2}+p_{\mathrm{add}}+|U|^2U=0, \tag{1}$$

where $p_{\mathrm{add}}=i\varepsilon\partial^3U/\partial\zeta^3$ for the former case or $p_{\mathrm{add}}=\varepsilon\partial^4U/\partial\zeta^4$ for the latter case and $|\varepsilon|\ll 1$. If $\varepsilon=0$, then Eq. (1) reduces to the conventional NLS equation that has the well-known family of one-soliton solutions:

$$U_0(\tau,\zeta)=\frac{\sqrt{2\alpha}}{\cosh[\sqrt{\alpha}(\zeta-v\tau)]}e^{i(\alpha\tau-v^2\tau/4+v\zeta/2)}. \tag{2}$$

These solitons exist for any $\alpha>0$. For simplicity we demonstrate the idea of resonant interactions between a soliton and linear waves for the case of the stationary soliton (taking $v=0$); however, a solution with any value of v can be easily considered as well.

As a first step in our search for the resonance we should make a change of variables in Eq. (1) in the form $U(\tau,\zeta)=W(\tau,\zeta)e^{i\alpha\tau}$ and obtain the equation for $W(\tau,\zeta)$,

$$i\frac{\partial W}{\partial \tau}+\frac{\partial^2 W}{\partial \zeta^2}-\alpha W+|W|^2W+p_{\mathrm{add}}(W)=0. \tag{3}$$

It is easy to see that

$$W_0(\zeta)=\frac{\sqrt{2\alpha}}{\cosh(\sqrt{\alpha}\zeta)} \tag{4}$$

is a stationary soliton solution of Eq. (3) at $p_{\mathrm{add}}=0$. The phase velocity of any spectral component of the soliton (4) is equal to zero. Now the condition for the phase velocity resonance of the soliton (4) with linear waves of Eq. (3) has the form $V(k) = -\alpha/k - k = 0$ [$V(k)$ stands for the phase velocity of a linear wave with wave number k], which shows that there is no resonance between any soliton (4) and any linear wave. However, the situation is different if the third-order dispersion $p_{\rm add} = i\varepsilon\partial^3 W/\partial\zeta^3$ is taken into account. Indeed, in this case the equation for the phase velocity resonance has the form $\varepsilon k^2 - \alpha/k - k = 0$ and the resonance between the soliton (4) and linear waves with wave numbers $k \approx 1/\varepsilon$ always takes place. This resonance results in radiation "beyond all orders" [8,10,11]. For the case of the fourth-order dispersion $p_{\rm add} = \varepsilon\partial^4 W/\partial\zeta^4$, in Eq. (3) two different situations are possible. Now the equation for the phase velocity resonance has the form $\varepsilon k^3 - \alpha/k - k = 0$. For $\varepsilon < 0$ this equation has no solutions, i.e., there is no resonance of linear waves with solitons, so that the latter still exist as stable stationary localized solutions [15,16]. For $\varepsilon > 0$ the situation is similar to the case of the third-order dispersion and the resonance always takes place, i.e., any single soliton radiates. Similar examples for the case of a generalized KdV equation with a small additional fifth-order derivative term have been analyzed earlier [8,17,18].

In the present paper we consider the case when a Hamiltonian perturbation to an integrable equation leads to the resonance of solitons with linear waves. Single solitons as stationary solutions do not survive this kind of perturbation and they always emit radiation, which, however, can be exponentially small (beyond all orders). Now the question is whether other types of stationary localized solutions can exist in such systems.

Recently some attempts to construct stationary localized solutions for the generalized NLS equation (1) with an additional small third-order derivative term were made [19] and a similar idea was also discussed in [20]. The approach of [19,20] is based on the idea that two radiative solitons can form a bound state as a result of mutual trapping of their own radiation. However, in Ref. [19] radiation is emitted only from one side of a soliton. This makes it impossible to construct *stationary* two-soliton solutions since, even after suppression of radiation in the asymptotic regions, a power flow from one soliton to the other still exists and the constructed solution is always effectively nonstationary.

Thus the question remains: Can stationary localized solutions exist being in resonance with linear waves? We will show below that the answer to this question can be positive if single solitonlike solutions of a problem radiate symmetrically in both directions from the soliton core (see Fig. 1). The main model that is considered in this paper is the dynamical equation

$$i\frac{\partial U}{\partial \tau} + \frac{\partial^2 U}{\partial \zeta^2} + \frac{\partial^4 U}{\partial \zeta^4} + |U|^2 U = 0, \tag{5}$$

which can be obtained from Eq. (1) (with an additional small positive fourth-order derivative term) by an exact scaling transformation [15]. Equation (5) describes a special regime of temporal pulse propagation in a nonlinear optical fiber; U stands for an envelope of a normalized electric field of an optical pulse. Equation (5) also has many nonoptical applications. For example, it describes propagation of "whistlers" in plasmas [21].

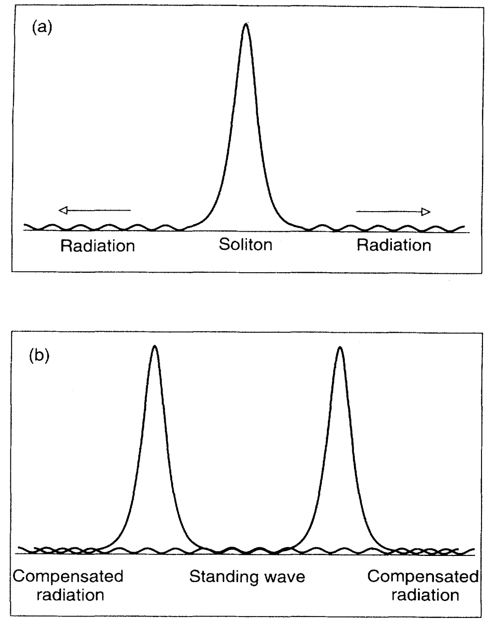

FIG. 1. (a) Schematic picture of a single symmetrically radiating soliton. (b) Bound state of two identical symmetrically radiating solitons. The first stage of stationary solution construction.

We also present the similar results concerning the other physically important system:

$$\begin{aligned} &i\frac{\partial w}{\partial \tau} + r\frac{\partial^2 w}{\partial \zeta^2} - w + w^* v = 0, \\ &i\sigma\frac{\partial v}{\partial \tau} + s\frac{\partial^2 v}{\partial \zeta^2} - \alpha v + \frac{1}{2}w^2 = 0. \end{aligned} \tag{6}$$

Equations (6) form a fundamental system describing two-wave interactions in dispersive (or diffractive) $\chi^{(2)}$ materials (see, e.g., [22,23]). In Eqs. (6) the functions w and v stand for envelopes of the normalized fundamental and second harmonics, respectively, σ ($\sigma > 0$) and α are continuous parameters, and r and s can be equal to 1 or -1 (for details see [22,23]). We note that the system (6) has other physical applications. For example, its stationary solutions can have the same form as those for the equations describing coupled Langmuir and ion-sound oscillations in plasma [24,25]. Note that the system (6) can be also considered as a perturbed integrable model since in the limit $\alpha \gg 1$ it can be formally reduced to a single NLS equation [23].

It is quite interesting to note that some exact solutions are known for both models (5) and (6) in an explicit analytical form. For example, Eq. (5) has the stationary solution [see Fig. 2(a)]

$$U(\zeta,\tau) \equiv u(\zeta)\, e^{i\alpha\tau} = \sqrt{\frac{6}{5}}\, \frac{\sinh(\zeta/\sqrt{10})}{\cosh^2(\zeta/\sqrt{10})} e^{i\frac{11}{100}\tau}, \quad (7)$$

which was found in [14].

The system (6) also admits exact solutions that describe two-wave parametric solitons in dispersive $\chi^{(2)}$ nonlinear media. For $r = 1$, $s = -1$, and $\alpha = 2$ it has the exact solution found in [26] [see Fig. 2(b)],

$$w = \frac{6\sqrt{2}\sinh\zeta}{\cosh^2\zeta}, \quad v = \frac{6}{\cosh^2\zeta}. \quad (8)$$

For $r = -1$, $s = -1$, and $\alpha = 1$, Eqs. (6) have another analytic solution, found in [27] [see Fig. 2(c)],

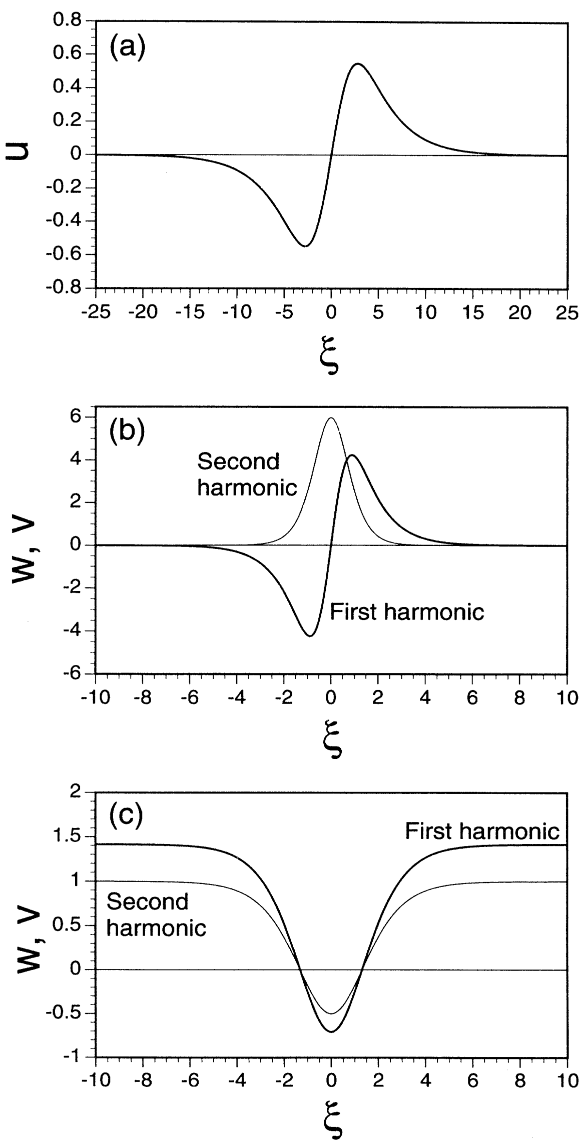

FIG. 2. Three "remarkable" exact solutions: (a) the solution (7) [14], (b) the solution (8) [26], and (c) the solution (9) [27].

$$w = \sqrt{2}\left[1 - \frac{3/2}{\cosh^2(\zeta/2)}\right], \qquad v = \left[1 - \frac{3/2}{\cosh^2(\zeta/2)}\right]. \quad (9)$$

We will show below that these three exact solutions (7)–(9) are the particular (simplest) representatives of a broad class of localized solutions, the so-called two-soliton radiationless bound states (BS's) of single radiating solitons, a fact that has been missed in the literature. Each of these BS's is formed with the help of a standing wave of trapped radiation that self-consistently produces local extrema in an effective interaction potential of two radiative solitons, which support this standing wave between them.

The remainder of the paper is organized as follows. In Sec. II we provide the outline of the analytic solution of the BS problem for the case of Eq. (5) and present corresponding numerical results. We also consider the stability of the discovered class of two-soliton BS's. In Sec. III we present results concerning BS's of radiating solitons of the system (6). Finally, Sec. IV concludes the paper.

II. BOUND STATES OF RADIATING SOLITONS

The physical picture for the formation of the soliton BS's due to trapping of radiation will be explained while analyzing Eq. (5), but it is also possible to carry out a similar analysis for the case of Eqs. (6). We start our analysis with a quick overview of the results related to the conventional NLS equation

$$i\frac{\partial U}{\partial \tau} + \frac{\partial^2 U}{\partial \zeta^2} + |U|^2 U = 0. \quad (10)$$

The NLS equation (10) can be exactly solved by means of the inverse scattering transform technique [28]. Equation (10) has a continuous family of stationary one-soliton solutions

$$U_0(\zeta,\tau) = \frac{\sqrt{2\alpha}}{\cosh[\sqrt{\alpha}(\zeta-\zeta_0)]} e^{i\alpha\tau + i\psi_0}, \quad (11)$$

where ζ_0 is an initial soliton center position and ψ_0 is an initial phase. Adding the fourth-order derivative term to the NLS equation, we obtain Eq. (5). The conventional solitons (11) are no longer stationary solutions of Eq. (5). Moreover, any localized one-hump pulse launched as an initial condition to Eq. (5) will radiate, i.e., stationary single solitons do not exist for Eq. (5) in principle. A formal analysis of the radiation problem for Eq. (5) can be found in [12,13]. This analysis shows that the solution (11) taken as an initial condition for Eq. (5) emits radiation on the wave number corresponding to the phase velocity resonance,

$$k_0 = \pm\sqrt{\frac{1+\sqrt{1+4\alpha}}{2}}. \quad (12)$$

The amplitude of the radiative waves can also be estimated [12,13]

$$a_{\text{rad}} \sim e^{-\frac{k_0 \pi}{2\sqrt{\alpha}}}. \tag{13}$$

In contrast to the case of the generalized NLS equation with an additional third-order derivative term [8–11,13], in the case of Eq. (5) the radiation is emitted symmetrically from both sides of the pulse. For $\sqrt{\alpha} \ll 1$ the intensity of radiation is exponentially small (beyond all orders), so that the soliton solution (11) is a good *approximate* solution of Eq. (5).

Now, as the first step to construct BS's of two identical radiative solitons, we need to locate them relative to each other in such a way that the interference pattern of their radiation vanishes exactly in both asymptotic regions $\zeta \to \pm\infty$ (see also [19]). Figure 1 illustrates this idea schematically. Without loss of generality we can assume that we have one soliton (11) with $\zeta_0 = 0$ and $\psi_0 = 0$, which we denote as $U_0(\zeta,\tau)$, and the other soliton $U_0(\zeta+\Delta\zeta,\tau)e^{i\Delta\psi}$ with the maximum at $\zeta_0 = \Delta\zeta > 0$ and the initial phase $\psi_0 = \Delta\psi > 0$. Both solitons emit radiation symmetrically and for each soliton taken separately there is a phase shift $\phi_0(\alpha)$ between the soliton and the emitted radiation [in general, $\phi_0(\alpha)$ is nonzero]. Following the approach of [20], it is possible to show that the radiation in the region $\zeta \to -\infty$ can be cancel if the radiation from the second soliton $[U_0(\zeta+\Delta\zeta,\tau)e^{i\Delta\psi}]$ is coming to the first soliton $[U_0(\zeta,\tau)]$ with the relative phase shift $-\phi_0(\alpha)$. Thus, in order to cancel the radiation we need to satisfy the condition

$$k_0\Delta\zeta + \Delta\psi + 2\phi_0(\alpha) = 2\pi n, \tag{14}$$

where $n = 1,2,3,\ldots$. However, the condition (14) does not guarantee the radiation compensation in the asymptotic region $\zeta \to \infty$. To cancel the radiation there we should satisfy the other condition that can be obtained in the similar way:

$$k_0\Delta\zeta - \Delta\psi + 2\phi_0(\alpha) = 2\pi m, \tag{15}$$

where $m = 0,1,2,3,\ldots$. It immediately follows from (14) and (15) that radiation can be canceled in both asymptotic regions only if two identical solitons are either in phase ($\Delta\psi = 0$) or completely out of phase ($\Delta\psi = \pi$). The form of the resulting two-soliton structure U_{TS} with the radiation canceled in both asymptotic regions (note that, in general, this two-soliton structure can still be *nonstationary*) is given by

$$U_{\text{TS}} = U_0(\zeta,\tau) + U_0(\zeta+\Delta\zeta,\tau)e^{i\Delta\psi} + f(\zeta,\tau,\Delta\zeta,\Delta\psi), \tag{16}$$

where $\Delta\psi = 0$ or π and f represents the standing wave formed by the trapped radiation (see Fig. 3 for examples). It is important to note that the standing wave f (in contrast to the radiation emitted by a single soliton) has *a zero* phase shift with respect to the solitons $U_0(\zeta,\tau)$ and $U_0(\zeta+\Delta\zeta,\tau)$. We can also consider the value of the discrete parameter n of Eq. (14) as the order of the standing wave f (it is related to the number of zeros in the central part of f). In general, it is quite complicated to find ϕ_0 and especially f analytically for $\sqrt{\alpha} \ll 1$ (i.e., in the approximation of weak radiation). Moreover, even the asymptotic result is hard to obtain, e.g., four known approaches for the calculation of the intensity of the asymptotic radiation for the case of the NLS equation with the third-order dispersion give four slightly different analytic results [8,10,11,13]. Since in this paper we are interested in the physical picture rather than in rigorous mathematical details, we will proceed with our analysis assuming that we already know f and ϕ_0 somehow.

Now to find *stationary* two-soliton BS's we can employ the effective particle approach of Ref. [29] (for the recent examples of using this approach see [16,20]). The effective particle approach shows that the interaction between two solitons in a Hamiltonian system is determined by the nonlinear part of the corresponding system Hamiltonian, which in the case of Eq. (5) is

$$H_{\text{int}} = -\frac{1}{2}\int_{-\infty}^{\infty} |U|^4 d\zeta. \tag{17}$$

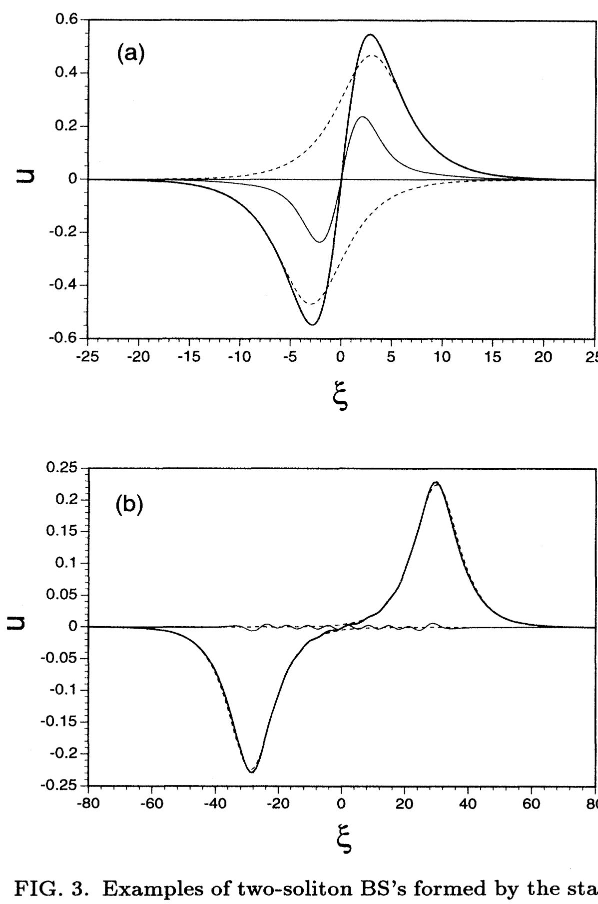

FIG. 3. Examples of two-soliton BS's formed by the standing wave of trapped radiation. Two-soliton BS's are shown by thick curves. Partial single solitons are shown by thin dashed curves and standing waves formed due to trapping of radiation are shown by thin solid curves. (a) The two-soliton BS at $n = 1$. (b) The two-soliton BS at $n = 9$.

Using the approximation of well-separated solitons, one obtains [20,29]

$$H_{\rm int}(\Delta\zeta,\Delta\psi) = -2\int_{-\infty}^{\infty} |U_1|^2 {\rm Re}[U_1(U_2^* + f^*)]d\zeta + (1 \leftrightarrow 2), \tag{18}$$

where the expression describing the interaction of the first single soliton with the tail of the second one and the standing wave of trapped radiation is written down explicitly. Because of the symmetry, a similar expression $(1 \leftrightarrow 2)$ has to be added to describe the interaction of the second soliton with the tail of the first one and the other edge of the standing wave. The effective interaction potential $H_{\rm int}$ [defined by (18)] depends on the relative distance $\Delta\zeta$ between the centers of the two solitons and their relative phase difference $\Delta\psi$. Two-soliton BS's exist if this effective interaction potential (18) has local extrema. These extrema are determined by the equations

$$\frac{\partial H_{\rm int}}{\partial \Delta\psi} = 0, \quad \frac{\partial H_{\rm int}}{\partial \Delta\zeta} = 0. \tag{19}$$

It is possible to show that the first of Eqs. (19) gives the result $\Delta\psi = 0$ or π, which has already been obtained from the conditions of trapped radiation (14) and (15). Thus we have only one variable $\Delta\zeta$ to be determined, but two equations: the condition of the trapped radiation (14) [or (15)] and the second equation of the system (19). For fixed values of the soliton parameter α, n (or m), and $\Delta\psi$ ($= 0$ or π) this system is overdetermined with respect to $\Delta\zeta$. However, if we solve this system assuming that two continuous variables $\Delta\zeta$ and α are unknown, we have a good chance to find *a discrete set* of the solutions $\Delta\zeta_n$ and α_n at least for some values of n and $\Delta\psi$.

As we have shown above, the difficulties in the analytic calculation of $\Delta\zeta_n$ and α_n are significant. However, there is a well-known and straightforward way to find the corresponding stationary solutions numerically, using the fact that possible two-soliton stationary solutions formed due to trapped radiation have flat phase fronts. Looking for the stationary solutions of Eq. (5) in the form $U(\zeta,\tau) = u(\zeta)e^{i\alpha\tau}$ (where u is a real function) we get the equation of real parameters

$$-\alpha u + \frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^4 u}{\partial \zeta^4} + u^3 = 0, \tag{20}$$

localized solutions of which correspond to stationary soliton solutions of Eq. (5) and can be found by means of the standard shooting technique (see, e.g., [15]). The results of this numerical analysis are shown in the form of the energy-dispersion diagram of Fig. 4, where Q is the energy of two-soliton BS's determined by

$$Q = \int_{-\infty}^{\infty} u^2 d\zeta = \int_{-\infty}^{\infty} |U|^2 d\zeta, \tag{21}$$

which is an integral of motion of Eq. (5).

One can see that, in addition to the already known exact analytic solution (7) (shown in Fig. 4 as a filled circle), which is a two-soliton BS of first order (i.e., the order n of the corresponding standing wave f is equal to one), there are also other two-soliton BS's of higher order. (The two-soliton BS's of third and ninth orders are explicitly shown in the bottom part of Fig. 4. The standing wave, formed by the trapped radiation, can be clearly seen between two soliton peaks.) Each of these solutions corresponds to some particular value of n of Eq. (14) and, in agreement with the above analysis, exists only for one unique value of the parameter α_n. We checked the validity of the relation (14) for these two-soliton BS's and found very good agreement between numerical and analytic results.

The question of stability of two-soliton and multisoliton BS's formed due to trapping of radiation is closely related to the question of their origin. The analysis of this section shows that these BS's are formed as a result of very delicate self-consistent balance, which can be easily broken. This itself does not prove that BS's formed due to trapping of radiation should be unstable. However, the fact that these BS's do not form continuous families in α strongly supports the idea of inherent instability of all BS's of radiating solitons. Indeed, any perturbation of a BS formed due to trapping of radiation will lead to

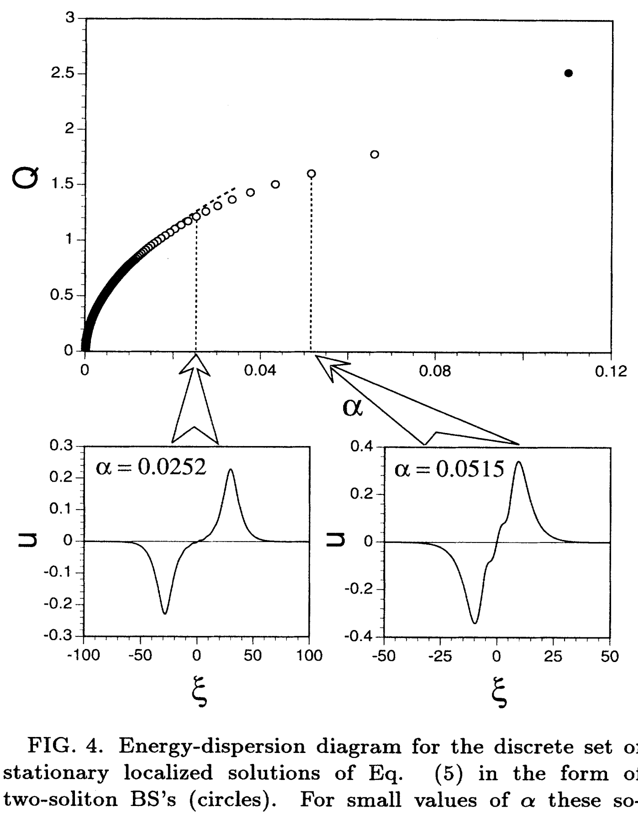

FIG. 4. Energy-dispersion diagram for the discrete set of stationary localized solutions of Eq. (5) in the form of two-soliton BS's (circles). For small values of α these solutions look as two NLS solitons (11) with a standing wave of trapped radiation between them. The filled circle corresponds to the exact solution (7) shown in Fig. 2(a). The dashed curve corresponds to the NLS equation limit. There is a discrete set of infinitely many two-soliton BS's for $\alpha \to 0$, but some of the corresponding opened circles are located very close to each other and cannot be distinguished in the scale of this figure.

an escape of some portion of radiation from this BS so that the value of the energy Q for the localized part of the solution will become lower. Since every BS formed due to trapping of radiation is not a representative of a continuous family, but instead a unique point in the corresponding energy-dispersion diagram, its shape cannot be easily adjusted to a new stationary state because there is no any other stationary solution with a slightly smaller value of energy. In principle, the initial BS can finally evolve into a BS of another order n with $Q_{\text{new}} < Q_{\text{init}}$, but since all BS states of the same class have shapes that are quite different from each other, even this, the most "successful," scenario of the soliton evolution shows that a small perturbation causes significant reshaping, which in turn means that the initial BS formed due to trapping of radiation is unstable.

In addition to these speculations we have carried out numerical simulations to investigate the stability of the discovered class of two-soliton BS's formed due to trapping of radiation. Except for the straightforward propagation of various slightly perturbed stationary BS's, we also studied the evolution of perturbation eigenmodes. To do this, we linearized equations around the BS of interest and solved the resulting linearized equation to find exponentially growing modes (for details of this technique see, for example, [30]). The results can be formulated in a simple way: *all BS's that we found are unstable.* The exponential increment of the fastest growing instability mode is higher for BS's of lower order and becomes very small for BS's of sufficiently high order. However, it is *always* positive. An example of evolution of a slightly perturbed solution (7) is shown in Fig. 5. A rigorous analytic analysis of the stability of soliton BS's formed by mutual trapping of the radiation is currently being carried out. [Note that we cannot use the stability criterion of the effective particlelike approach in a straightforward way (as in [20]), since it does not give a sufficient condition of stability for nonintegrable systems [16].]

III. OTHER EXAMPLES

We repeat the similar analytic and numerical analysis for the system (6) and found that, in addition to two-

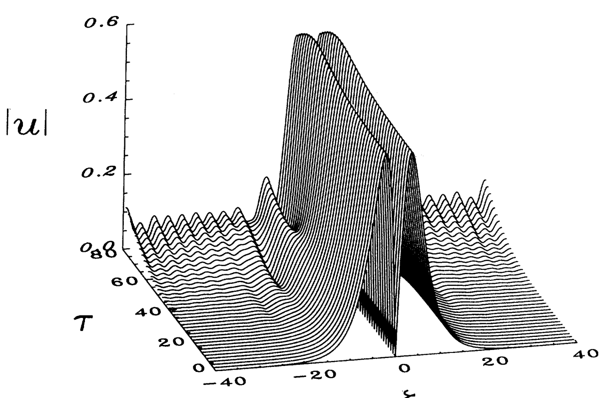

FIG. 5. Evolution of a slightly perturbed solution (7) (two-soliton BS of first order).

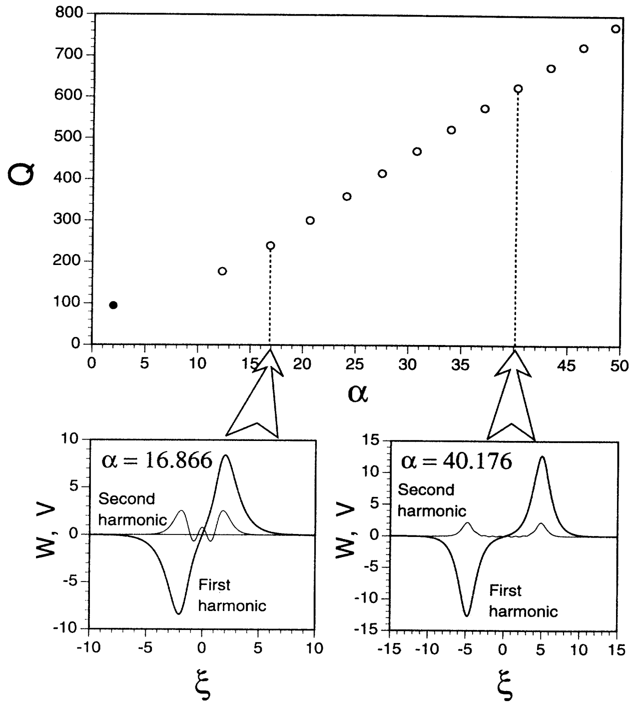

FIG. 6. Energy-dispersion diagram for the discrete set of stationary localized solutions of the system (6) with $r = 1$, $s = -1$ in the form of bright two-soliton BS's (circles). The filled circle corresponds to the exact solution (8) shown in Fig. 2(b).

soliton BS's of first order (8) and (9), BS's of radiating solitons of higher orders exist as well. (For the theory of radiating dark solitons see, e.g., [31].) They are presented in Figs. 6 and 7 for the cases $r = 1$, $s = -1$ and $r = -1$, $s = -1$ in Eqs. (6), respectively. In the energy-dispersion diagram of Fig. 6 the function Q is the total energy of two-wave bright BS's, which is determined by

$$Q = \int_{-\infty}^{\infty} (|w|^2 + |v|^2) d\zeta. \tag{22}$$

In the energy-dispersion diagram of Fig. 7 the function Q_c is the complimentary energy of two-wave dark BS's, which is determined by

$$Q_c = \int_{-\infty}^{\infty} (2\alpha - |w|^2 + 1 - |v|^2) d\zeta. \tag{23}$$

These Q and Q_c are integrals of motion of the corresponding versions of the system (6) for some particular choice of the parameter σ ($\sigma = 1/2$). The standing wave, formed due to trapping of radiation, is also clearly seen between single two-wave solitons (see the bottom parts of Figs. 6 and 7). Note that Figs. 4, 6, and 7 show not all found stationary solutions of the corresponding equations, but only two-soliton BS's. Multisoliton BS's formed due to trapping of radiation exist as well and they also have been found numerically. Finally, we checked numerically the stability of all classes of radiating soliton

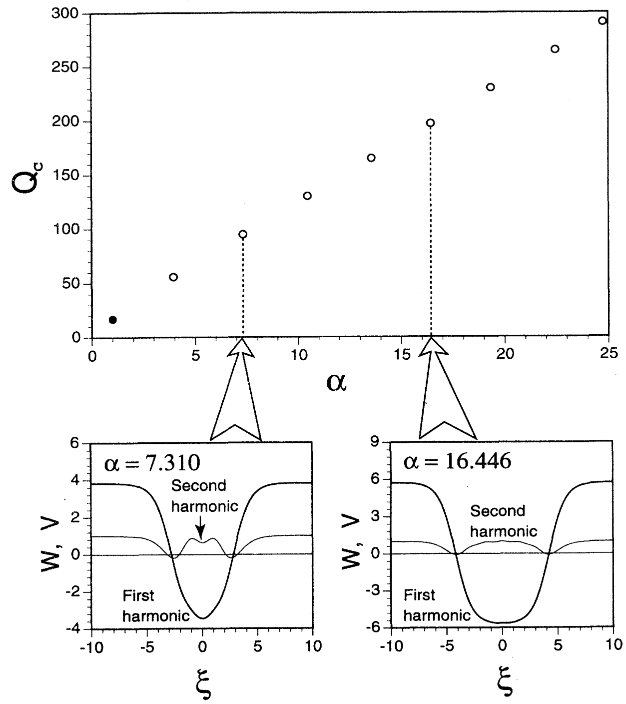

FIG. 7. Energy-dispersion diagram for the discrete set of stationary localized solutions of the system (6) with $r = -1$, $s = -1$ in the form of dark two-soliton BS's (circles). The filled circle corresponds to the exact solution (9) shown in Fig. 2(c).

BS's of the system (6). Again all these BS's are found to be unstable.

IV. CONCLUSIONS

We have presented constructive examples that for some dynamical systems stationary two-soliton (and multisoliton) solutions can exist being in resonance with small amplitude linear waves. The necessary condition for the existence of such BS's of radiating solitons is a symmetric pattern of radiation emitted from the soliton core of each single pulse. It is very important that, in contrast to conventional solitons, soliton bound states formed due to trapping of radiation do not form continuous families of localized solutions. As a consequence, all BS's of radiating solitons have no chance to be stable.

We propose and prove (analytically and numerically) that the exact solutions (7)–(9) are, in fact, two-soliton BS's formed due to trapping of radiation. We also found three corresponding classes of other two-soliton solutions formed due to trapping of radiation. Representatives of these classes can be classified by the order n of the standing wave of trapped radiation formed by interacting single solitons. The stability problem for all three classes of the found solitons has also been considered.

This paper provides insight into the physical essence of broad classes of soliton solutions. In the works where the exact solutions (7)–(9) were obtained, nothing was mentioned about the origin and the physical nature of these solutions. Moreover, some misunderstanding and mistakes that occurred in the literature concerning this topic would have been avoided had the physical picture discussed in the present study been known. For example, in the paper [27] it was claimed that the solution (9) is *"stable."* In [25] some solutions analogous to two-soliton and multisoliton BS's of our paper were declared to be *"rather stable."* Finally, in [19] it was mentioned that the two-hump solutions, formed by radiative solitons, can be *"quasistable"* in the sense that their instability growth rate is small. Formally the last statement is correct since a two-soliton BS of sufficiently high order does not have fast growing instability modes. However, in this case the bound energy (18) for such a BS is also exponentially small since there is practically no radiation emitted. Thus, in principle, we do not have a quasistable two-soliton BS, but two quasistable single solitons, which nearly do not interact with each other.

Finally, we would like to note that the approach and results obtained in the present paper can be readily applied to other models of a different physical context where radiative solitons exist. As a matter of fact, this can be expected in many other physical models where stationary solitonlike solutions are described by nonintegrable dynamical systems with a rather complicated behavior of the separatrix phase trajectories near critical points.

ACKNOWLEDGMENTS

The author is grateful to Dr. Yu. S. Kivshar and Dr. N. N. Akhmediev for fruitful discussions. This work was supported by the Australian Photonics Cooperative Research Centre.

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